

# On regular reduced products\*

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## Abstract

Assume  $\langle \aleph_0, \aleph_1 \rangle \rightarrow \langle \lambda, \lambda^+ \rangle$ . Assume  $M$  is a model of a first order theory  $T$  of cardinality at most  $\lambda^+$  in a vocabulary  $\mathcal{L}(T)$  of cardinality  $\leq \lambda$ . Let  $N$  be a model with the same vocabulary. Let  $\Delta$  be a set of first order formulas in  $\mathcal{L}(T)$  and let  $D$  be a regular filter on  $\lambda$ . Then  $M$  is  $\Delta$ -embeddable into the reduced power  $N^\lambda/D$ , provided that every  $\Delta$ -existential formula true in  $M$  is true also in  $N$ . We obtain the following corollary: for  $M$  as above and  $D$  a regular ultrafilter over  $\lambda$ ,  $M^\lambda/D$  is  $\lambda^{++}$ -universal. Our second result is as follows: For  $i < \mu$  let  $M_i$  and  $N_i$  be elementarily equivalent models of a vocabulary which has cardinality  $\leq \lambda$ . Suppose  $D$  is a regular filter on  $\mu$  and  $\langle \aleph_0, \aleph_1 \rangle \rightarrow \langle \lambda, \lambda^+ \rangle$  holds. We show that then the second player has a winning strategy in the Ehrenfeucht-Fraïssé game of length  $\lambda^+$  on  $\prod_i M_i/D$  and  $\prod_i N_i/D$ . This yields the following corollary: Assume GCH and  $\lambda$  regular (or just  $\langle \aleph_0, \aleph_1 \rangle \rightarrow \langle \lambda, \lambda^+ \rangle$  and  $2^\lambda = \lambda^+$ ). For  $L$ ,  $M_i$  and  $N_i$  be as above, if  $D$  is a regular filter on  $\lambda$ , then  $\prod_i M_i/D \cong \prod_i N_i/D$ .

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# 1 Introduction

Suppose  $M$  is a first order structure and  $F$  is the Frechet filter on  $\omega$ . Then the reduced power  $M^\omega/F$  is  $\aleph_1$ -saturated and hence  $\aleph_2$ -universal ([6]). This was generalized by Shelah in [10] to any filter  $F$  on  $\omega$  for which  $B^\omega/F$  is  $\aleph_1$ -saturated, where  $B$  is the two element Boolean algebra, and in [8] to all regular filters on  $\omega$ . In the first part of this paper we use the combinatorial principle  $\square_\lambda^{b*}$  of Shelah [11] to generalize the result from  $\omega$  to arbitrary  $\lambda$ , assuming  $\langle \aleph_0, \aleph_1 \rangle \rightarrow \langle \lambda, \lambda^+ \rangle$ . This gives a partial solution to Conjecture 19 in [3]: if  $D$  is a regular ultrafilter over  $\lambda$ , then for all infinite  $M$ , the ultrapower  $M^\lambda/D$  is  $\lambda^{++}$ -universal.

The second part of this paper addresses Problem 18 in [3], which asks if it is true that if  $D$  is a regular ultrafilter over  $\lambda$ , then for all elementarily equivalent models  $M$  and  $N$  of cardinality  $\leq \lambda$  in a vocabulary of cardinality  $\leq \lambda$ , the ultrapowers  $M^\lambda/D$  and  $N^\lambda/D$  are isomorphic. Keisler [7] proved this for good  $D$  assuming  $2^\lambda = \lambda^+$ . Benda [1] weakened "good" to "contains a good filter". We prove the claim in full generality, assuming  $2^\lambda = \lambda^+$  and  $\langle \aleph_0, \aleph_1 \rangle \rightarrow \langle \lambda, \lambda^+ \rangle$ .

Regarding our assumption  $\langle \aleph_0, \aleph_1 \rangle \rightarrow \langle \lambda, \lambda^+ \rangle$ , by Chang's Two-Cardinal Theorem ([2])  $\langle \aleph_0, \aleph_1 \rangle \rightarrow \langle \lambda, \lambda^+ \rangle$  is a consequence of  $\lambda = \lambda^{<\lambda}$ . So our Theorem 2 settles Conjecture 19 of [3], and Theorem 13 settles Conjecture 18 of [3], under GCH for  $\lambda$  regular. For singular strong limit cardinals  $\langle \aleph_0, \aleph_1 \rangle \rightarrow \langle \lambda, \lambda^+ \rangle$  follows from  $\square_\lambda$  (Jensen [5]). In the so-called Mitchell's model ([9])  $\langle \aleph_0, \aleph_1 \rangle \not\rightarrow \langle \aleph_1, \aleph_2 \rangle$ , so our assumption is independent of ZFC.

# 2 Universality

**Definition 1** *Suppose  $\Delta$  is a set of first order formulas of vocabulary  $L$ . The set of  $\Delta$ -existential formulas is the set of formulas of the form*

$$\exists x_1 \dots \exists x_n (\phi_1 \wedge \dots \wedge \phi_n),$$

*where each  $\phi_i$  is in  $\Delta$ . The set of weakly  $\Delta$ -existential formulas is the set of formulas of the above form, where each  $\phi_i$  is in  $\Delta$  or is the negation of a formula in  $\Delta$ . If  $M$  and  $N$  are  $L$ -structures and  $h : M \rightarrow N$ , we say that  $h$  is a  $\Delta$ -homomorphism if  $h$  preserves the truth of  $\Delta$ -formulas. If  $h$  preserves also the truth of negations of  $\Delta$ -formulas, it is called a  $\Delta$ -embedding.*

**Theorem 2** Assume  $\langle \aleph_0, \aleph_1 \rangle \rightarrow \langle \lambda, \lambda^+ \rangle$ . Let  $M$  be a model of a first order theory  $T$  of cardinality at most  $\lambda^+$ , in a language  $L$  of cardinality  $\leq \lambda$  and let  $N$  be a model with the same vocabulary. Let  $\Delta$  be a set of first order formulas in  $L$  and let  $D$  be a regular filter on  $\lambda$ . We assume that every weakly  $\Delta$ -existential sentence true in  $M$  is true also in  $N$ . Then there is a  $\Delta$ -embedding of  $M$  into the reduced power  $N^\lambda/D$ .

By letting  $\Delta$  be the set of all first order sentences, we get from Theorem 2 and Łoś' Lemma:

**Corollary 3** Assume  $\langle \aleph_0, \aleph_1 \rangle \rightarrow \langle \lambda, \lambda^+ \rangle$ . If  $M$  is a model with vocabulary  $\leq \lambda$ , and  $D$  is a regular ultrafilter over  $\lambda$ , then  $M^\lambda/D$  is  $\lambda^{++}$ -universal, i.e. if  $M'$  is of cardinality  $\leq \lambda^+$ , and  $M' \equiv M$ , then  $M'$  is elementarily embeddable into the ultrapower  $M^\lambda/D$ .

We can replace "weakly  $\Delta$ -existential" by " $\Delta$ -existential" in the Theorem, if we only want a  $\Delta$ -homomorphism.

The proof of Theorem 2 is an induction over  $\lambda$  and  $\lambda^+$  respectively, as follows. Suppose  $M = \{a_\zeta : \zeta < \lambda^+\}$ . We associate to each  $\zeta < \lambda^+$  finite sets  $u_i^\zeta$ ,  $i < \lambda$ , and represent the formula set  $\Delta$  as a union of finite sets  $\Delta_i$ . At stage  $i$ , for each  $\zeta < \lambda^+$  we consider the  $\Delta_i$ -type of the elements  $a_\zeta$  of the model whose indices lie in the set  $u_i^\zeta$ ,  $\zeta < \lambda^+$ . This will yield a witness  $f_\zeta(i)$  in  $N$  at stage  $i, \zeta$ . Our embedding is then given by  $a_\zeta \mapsto \langle f_\zeta(i) : i < \lambda \rangle / D$ .

We need first an important lemma, reminiscent of Proposition 5.1 in [11]:

**Lemma 4** Assume  $\langle \aleph_0, \aleph_1 \rangle \rightarrow \langle \lambda, \lambda^+ \rangle$ . Let  $D$  be a regular filter on  $\lambda$ . There exist sets  $u_i^\zeta$  and integers  $n_i$  for each  $\zeta < \lambda^+$  and  $i < \lambda$  such that for each  $i, \zeta$

- (i)  $|u_i^\zeta| < n_i < \omega$
- (ii)  $u_i^\zeta \subseteq \zeta$
- (iii) Let  $B$  be a finite set of ordinals and let  $\zeta$  be such that  $B \subseteq \zeta < \lambda^+$ . Then  $\{i : B \subseteq u_i^\zeta\} \in D$
- (iv) Coherency:  $\gamma \in u_i^\zeta \Rightarrow u_i^\gamma = u_i^\zeta \cap \gamma$

Assuming the lemma, and letting  $M = \{a_\zeta : \zeta < \lambda^+\}$  we now define, for each  $\zeta$ , a function  $f_\zeta : \lambda \mapsto N$ .

Let  $\Delta = \{\phi_\alpha : \alpha < \lambda\}$  and let  $\{A_\alpha : \alpha < \lambda\}$  be a family witnessing the regularity of D. Thus for each  $i$ , the set  $w_i = \{\alpha : i \in A_\alpha\}$  is finite. Let  $\Delta_i = \{\phi_\alpha : \alpha \in w_i\}$ , and let  $u_i^\zeta, n_i$  be as in the lemma.

We define a sequence of formulas essential to the proof: suppose  $\zeta < \lambda^+$  and  $i < \lambda$ . Let  $m_i^\zeta = |u_i^\zeta|$  and let

$$u_i^\zeta = \{\xi_{\zeta,i,0}, \dots, \xi_{\zeta,i,m_i^\zeta-1}\}$$

be an increasing enumeration of  $u_i^\zeta$ . (We adopt henceforth the convention that any enumeration of  $u_i^\zeta$  that is given is an increasing enumeration.) Let  $\bar{\theta}_i^\zeta$  be the  $\Delta_i$ -type of the tuple  $\langle a_{\xi_{\zeta,i,0}}, \dots, a_{\xi_{\zeta,i,m_i^\zeta-1}}, a_\zeta \rangle$  in  $M$ . (So every  $\phi(x_{\xi_{\zeta,i,0}}, \dots, x_{\xi_{\zeta,i,m_i^\zeta-1}}, x_\zeta) \in \Delta_i$  or its negation occurs as a conjunct of  $\bar{\theta}_i^\zeta$ , according to whether  $\phi(a_{\xi_{\zeta,i,0}}, \dots, a_{\xi_{\zeta,i,m_i^\zeta-1}}, a_\zeta)$  or  $\neg\phi(a_{\xi_{\zeta,i,0}}, \dots, a_{\xi_{\zeta,i,m_i^\zeta-1}}, a_\zeta)$  holds in  $M$ .) We define the formula  $\theta_i^\zeta$  for each  $i$  by downward induction on  $m_i^\zeta$  as follows:

Case 1:  $m_i^\zeta = n_i$ . Let  $\theta_i^\zeta = \bigwedge \bar{\theta}_i^\zeta$ .

Case 2:  $m_i^\zeta < n_i$ . Let  $\theta_i^\zeta$  be the conjunction of  $\bar{\theta}_i^\zeta$  and all formulas of the form  $\exists x_{m_i^\epsilon} \theta_i^\epsilon(x_0, \dots, x_{m_i^\epsilon-1}, x_{m_i^\epsilon})$ , where  $\epsilon$  satisfies  $u_i^\epsilon = u_i^\zeta \cup \{\zeta\}$  and hence  $m_i^\epsilon = m_i^\zeta + 1$ .

An easy induction shows that for a fixed  $i < \lambda$ , the cardinality of the set  $\{\theta_i^\zeta : \zeta < \lambda^+\}$  is finite, using  $n_i$ .

Let  $i < \lambda$  be fixed. We define  $f_\zeta(i)$  by induction on  $\zeta < \lambda^+$  in such a way that the following condition remains valid:

(IH) If  $\zeta^* < \zeta$  and  $u_i^{\zeta^*} = \{r_{\epsilon_1}, \dots, r_{\epsilon_k}\}$ , then  $N \models \theta_i^{\zeta^*}(f_{\epsilon_1}(i), \dots, f_{\epsilon_k}(i), f_{\zeta^*}(i))$ .

To define  $f_\zeta(i)$ , we consider different cases:

Case 1:  $n_i = m_i^\zeta$ .

Case 1.1:  $n_i = 0$ . Then  $\theta_i^\zeta$  is the  $\Delta_i$  type of the element  $a_\zeta$ . But then

$$\begin{aligned} M &\models \theta_i^\zeta(a_\zeta) \Rightarrow \\ M &\models \exists x_0 \theta_i^\zeta(x_0) \Rightarrow \\ N &\models \exists x_0 \theta_i^\zeta(x_0), \end{aligned}$$

where the last implication follows from the assumption that  $N$  satisfies the weakly  $\Delta$ -existential formulas holding in  $M$ . Now choose an element  $b \in N$  to witness this formula and set  $f_\zeta(i) = b$ .

Case 1.2:  $n_i > 0$ . Let  $u_i^\zeta = \{\xi_1, \dots, \xi_{m_i^\zeta}\}$ . Since  $m_i^\zeta = n_i$ , the formula  $\theta_i^\zeta$  is the  $\Delta_i$ -type of the elements  $\{\xi_1, \dots, \xi_{m_i^\zeta}\}$ . By assumption  $\gamma = \xi_{m_i^\zeta}$  is the maximum element of  $u_i^\zeta$ . Thus by coherency,  $u_i^\gamma = u_i^\zeta \cap \gamma = \{\xi_1, \dots, \xi_{m_i^\zeta-1}\}$ . Since  $\gamma < \zeta$ , we know by the induction hypothesis that

$$N \models \theta_i^\gamma(f_{\xi_1}(i), \dots, f_{\xi_{m_i^\zeta-1}}(i)).$$

By the formula construction  $\theta_i^\gamma$  contains the formula  $\exists x_{m_i^\gamma} \theta_i^\zeta(x_1, \dots, x_{m_i^\zeta})$ , since  $u_i^\zeta = u_i^\gamma \cup \{\gamma\}$  and since  $m_i^\gamma < n_i$ . Thus

$$N \models \exists x_{m_i^\zeta+1} \theta_i^\zeta(f_{\xi_1}(i), \dots, f_{\xi_{m_i^\zeta}}(i), x_{m_i^\zeta+1}).$$

As before choose an element  $b \in N$  to witness this formula and set  $f_\zeta(i) = b$ .

Case 2:  $m_i^\zeta < n_i$ . Let  $u_i^\zeta = \{\xi_1, \dots, \xi_{m_i^\zeta}\}$ . We have that  $M \models \theta_i^\zeta(a_{\xi_1}, \dots, a_{\xi_{m_i^\zeta}}, a_\zeta)$ , and therefore  $M \models \exists x_{m_i^\zeta+1} \theta_i^\zeta(a_{\xi_1}, \dots, a_{\xi_{m_i^\zeta}}, x_{m_i^\zeta+1})$ . Let  $\gamma = \max(u_i^\zeta) = \xi_{m_i^\zeta}$ .

By coherency,  $u_i^\gamma = u_i^\zeta \cap \gamma$  and therefore since  $\gamma < \zeta$  by the induction hypothesis we have that

$$N \models \theta_i^\gamma(f_{\xi_1}(i), \dots, f_{\xi_{m_i^\zeta-1}}(i), f_\gamma(i)).$$

But then as in case 1.2 we can infer that

$$N \models \exists x_{m_i^\zeta+1} \theta_i^\zeta(f_{\xi_1}(i), \dots, f_{\xi_{m_i^\zeta}}(i), x_{m_i^\zeta+1}).$$

As in case 1 choose an element  $b \in N$  to witness this formula and set  $f_\zeta(i) = b$ .

It remains to be shown that the mapping  $a_\zeta \mapsto \langle f_\zeta(i) : i < \lambda \rangle / D$  satisfies the requirements of the theorem, i.e. we must show, for all  $\phi$  which is in  $\Delta$ , or whose negation is in  $\Delta$ ,

$$M \models \phi(a_{\xi_1}, \dots, a_{\xi_k}) \Rightarrow \{i : N \models \phi(f_{\xi_1}(i), \dots, f_{\xi_k}(i))\} \in D.$$

So let such a  $\phi$  be given, and suppose  $M \models \phi(a_{\xi_1}, \dots, a_{\xi_k})$ . Let  $I_\phi = \{i : N \models \phi(f_{\xi_1}(i), \dots, f_{\xi_k}(i))\}$ . We wish to show that  $I_\phi \in D$ . Let  $\alpha < \lambda$  so that

$\phi$  is  $\phi_\alpha$  or its negation. It suffices to show that  $A_\alpha \subseteq I_\phi$ . Let  $\zeta < \lambda^+$  be such that  $\{\xi_1, \dots, \xi_n\} \subseteq \zeta$ . By Lemma 4 condition (iii),  $\{i : \{\xi_1, \dots, \xi_n\} \subseteq u_i^\zeta\} \in D$ . So it suffices to show

$$A_\alpha \cap \{i : \{\xi_1, \dots, \xi_n\} \subseteq u_i^\zeta\} \subseteq I_\phi.$$

Let  $i \in A_\alpha$  such that  $\{\xi_1, \dots, \xi_n\} \subseteq u_i^\zeta$ . By the definition of  $\theta_i^\zeta$  we know that  $N \models \theta_i^\zeta(f_{\xi_1}(i), \dots, f_{\xi_k}(i))$ . But the  $\Delta_i$ -type of the tuple  $\langle \xi_1, \dots, \xi_k \rangle$  occurs as a conjunct of  $\theta_i^\zeta$ , and therefore  $N \models \phi(f_{\xi_1}(i), \dots, f_{\xi_k}(i))$   $\square$

### 3 Proof of Lemma 4

We now prove Lemma 4. We first prove a weaker version in which the filter is not given in advance:

**Lemma 5** *Assume  $\langle \aleph_0, \aleph_1 \rangle \rightarrow \langle \lambda, \lambda^+ \rangle$ . There exist sets  $\langle u_i^\zeta : \zeta < \lambda^+, i < \text{cof}(\lambda) \rangle$ , integers  $n_i$  and a regular filter  $D$  on  $\lambda$ , generated by  $\lambda$  sets, such that (i)-(iv) of Lemma 4 hold.*

**Proof.** By [11, Proposition 5.1, p. 149] the assumption  $\langle \aleph_0, \aleph_1 \rangle \rightarrow \langle \lambda, \lambda^+ \rangle$  is equivalent to:

$\square_\lambda^{b^*}$  : There is a  $\lambda^+$ -like linear order  $L$ , sets  $\langle C_a^\zeta : a \in L, \zeta < \text{cf}(\lambda) \rangle$ , equivalence relations  $\langle E^\zeta : \zeta < \text{cf}(\lambda) \rangle$ , and functions  $\langle f_{a,b}^\zeta : \zeta < \lambda, a \in L, b \in L \rangle$  such that

- (i)  $\bigcup_\zeta C_a^\zeta = \{b : b <_L a\}$  (an increasing union in  $\zeta$ ).
- (ii) If  $b \in C_a^\zeta$ , then  $C_b^\zeta = \{c \in C_a^\zeta : c <_L b\}$ .
- (iii)  $E^\zeta$  is an equivalence relation on  $L$  with  $\leq \lambda$  equivalence classes.
- (iv) If  $\zeta < \xi < \text{cf}(\lambda)$ , then  $E^\xi$  refines  $E^\zeta$ .
- (v) If  $aE^\zeta b$ , then  $f_{a,b}^\zeta$  is an order-preserving one to one mapping from  $C_a^\zeta$  onto  $C_b^\zeta$  such that for  $d \in C_a^\zeta$ ,  $dE^\zeta f_{a,b}^\zeta(d)$ .
- (vi) If  $\zeta < \xi < \text{cf}(\lambda)$  and  $aE^\xi b$ , then  $f_{a,b}^\zeta \subseteq f_{a,b}^\xi$ .
- (vii) If  $f_{a,b}^\zeta(a_1) = b_1$ , then  $f_{a_1,b_1}^\zeta \subseteq f_{a,b}^\zeta$ .
- (viii) If  $a \in C_b^\zeta$  then  $\neg E^\zeta(a, b)$ .

This is not quite enough to prove Lemma 5, so we have to work a little more. Let

$$\Xi_\zeta = \{a/E^\zeta : a \in L\}.$$

We assume, for simplicity, that  $\zeta \neq \xi$  implies  $\Xi_\zeta \cap \Xi_\xi = \emptyset$ . Define for  $t_1, t_2 \in \Xi_\zeta$ :

$$t_1 <_\zeta t_2 \iff (\exists a_1 \in t_1)(\exists a_2 \in t_2)(a_1 \in C_{a_2}^\zeta).$$

**Proposition 6**  $\langle \Xi_\zeta, <_\zeta \rangle$  is a tree order with  $cf(\lambda)$  as the set of levels.

**Proof.** We need to show (a)  $t_1 <_\zeta t_2 <_\zeta t_3$  implies  $t_1 <_\zeta t_3$ , and (b)  $t_1 <_\zeta t_3$  and  $t_2 <_\zeta t_3$  implies  $t_1 <_\zeta t_2$  or  $t_2 <_\zeta t_1$  or  $t_1 = t_2$ . For the first,  $t_1 <_\zeta t_2$  implies there exists  $a_1 \in t_1$  and  $a_2 \in t_2$  such that  $a_1 \in C_{a_2}^\zeta$ . Similarly  $t_2 <_\zeta t_3$  implies there exists  $b_2 \in t_2$  and  $b_3 \in t_3$  such that  $b_2 \in C_{b_3}^\zeta$ . Now  $a_2 E^\zeta b_2$  and hence we have the order preserving map  $f_{a_2, b_2}^\zeta$  from  $C_{a_2}^\zeta$  onto  $C_{b_2}^\zeta$ . Recalling  $a_1 \in C_{a_2}^\zeta$ , let  $f_{a_2, b_2}^\zeta(a_1) = b_1$ . Then by (vi),  $a_1 E^\zeta b_1$  and hence  $b_1 \in t_1$ . But then  $b_1 \in C_{b_2}^\zeta$  implies  $b_1 \in C_{b_3}^\zeta$ , by coherence and the fact that  $b_2 \in C_{b_3}^\zeta$ . But then it follows that  $t_1 <_\zeta t_3$ .

Now assume  $t_1 <_\zeta t_3$  and  $t_2 <_\zeta t_3$ . Let  $a_1 \in t_1$  and  $a_3 \in t_3$  be such that  $a_1 \in C_{a_3}^\zeta$ , and similarly let  $b_2$  and  $b_3$  be such that  $b_2 \in C_{b_3}^\zeta$ .  $a_3 E^\zeta b_3$  implies we have the order preserving map  $f_{a_3, b_3}^\zeta$  from  $C_{a_3}^\zeta$  to  $C_{b_3}^\zeta$ . Letting  $f_{a_3, b_3}^\zeta(a_1) = b_1$ , we see that  $b_1 \in C_{b_3}^\zeta$ . If  $b_1 <_L b_2$ , then we have  $C_{b_2}^\zeta = C_{b_3}^\zeta \cap \{c \mid c < b_2\}$  which implies  $b_1 \in C_{b_2}^\zeta$ , since, as  $f_{a_3, b_3}^\zeta$  is order preserving,  $b_1 <_L b_2$ . Thus  $t_1 <_\zeta t_2$ . The case  $b_2 <_L b_1$  is proved similarly, and  $b_1 = b_2$  is trivial.  $\square$

For  $a <_L b$  let

$$\xi(a, b) = \min\{\zeta : a \in C_b^\zeta\}.$$

Denoting  $\xi(a, b)$  by  $\xi$ , let

$$tp(a, b) = \langle a/E^\xi, b/E^\xi \rangle.$$

If  $a_1 <_L \dots <_L a_n$ , let

$$tp(\langle a_1, \dots, a_n \rangle) = \{\langle l, m, tp(a_l, a_m) \rangle \mid 1 \leq l < m \leq n\}$$

and

$$\Gamma = \{tp(\vec{a}) : \vec{a} \in {}^{<\omega}L\}.$$

For  $t = tp(\vec{a})$ ,  $\vec{a} \in {}^n L$  we use  $n_t$  to denote the length of  $\vec{a}$ .

**Proposition 7** *If  $a_0 <_L \dots <_L a_n$ , then*

$$\max\{\xi(a_l, a_m) : 0 \leq l < m \leq n\} = \max\{\xi(a_l, a_n) : 0 \leq l < n\}.$$

**Proof.** Clearly the right hand side is  $\leq$  the left hand side. To show the left hand side is  $\leq$  the right hand side, let  $l < m < n$  be arbitrary. If  $\xi(a_l, a_n) \leq \xi(a_m, a_n)$ , then  $\xi(a_l, a_m) \leq \xi(a_m, a_n)$ . On the other hand, if  $\xi(a_l, a_n) > \xi(a_m, a_n)$ , then  $\xi(a_l, a_m) \leq \xi(a_l, a_n)$ . In either case  $\xi(a_l, a_m) \leq \max\{\xi(a_k, a_n) : 0 \leq k < n\}$ .  $\square$

Let us denote  $\max\{\xi(a_l, a_n) : 0 \leq l < n\}$  by  $\xi(\vec{a})$ . We define on  $\Gamma$  a two-place relation  $\leq_\Gamma$  as follows:

$$t_1 <_\Gamma t_2$$

if there exists a tuple  $\langle a_0, \dots, a_{n_{t_2}-1} \rangle$  realizing  $t_2$  such that some subsequence of the tuple realizes  $t_1$ .

Clearly,  $\langle \Gamma, \leq_\Gamma \rangle$  is a directed partial order.

**Proposition 8** *For  $t \in \Gamma$ ,  $t = tp(b_0, \dots, b_{n-1})$  and  $a \in L$ , there exists at most one  $k < n$  such that  $b_k E^{\xi(b_0, \dots, b_{n-1})} a$ .*

**Proof.** Let  $\zeta = \xi(b_0, \dots, b_{n-1})$  and let  $b_{k_1} \neq b_{k_2}$  be such that  $b_{k_1} E^\zeta a$  and  $b_{k_2} E^\zeta a$ ,  $k_1, k_2 \leq n-1$ . Without loss of generality, assume  $b_{k_1} < b_{k_2}$ . Since  $E^\zeta$  is an equivalence relation,  $b_{k_2} E^\zeta b_{k_1}$  and thus we have an order preserving map  $f_{b_{k_2}, b_{k_1}}^\zeta$  from  $C_{b_{k_2}}^\zeta$  to  $C_{b_{k_1}}^\zeta$ . Also  $b_{k_1} \in C_{b_{k_2}}^\zeta$ , by the definition of  $\zeta$  and by coherence, and therefore  $f_{b_{k_2}, b_{k_1}}^\zeta(b_{k_1}) E^\zeta b_{k_1}$ . But this contradicts (viii), since  $f_{b_{k_2}, b_{k_1}}^\zeta(b_{k_1}) \in C_{b_{k_1}}^\zeta$ .  $\square$

**Definition 9** *For  $t \in \Gamma$ ,  $t = tp(b_0, \dots, b_{n-1})$  and  $a \in L$  suppose there exists  $k < n$  such that  $b_k E^{\xi(b_0, \dots, b_{n-1})} a$ . Then let  $u_t^a = \{f_{a, b_k}^{\xi(b_0, \dots, b_{n-1})}(b_l) : l < k\}$ . Otherwise, let  $u_t^a = \emptyset$ .*

Finally, let  $D$  be the filter on  $\Gamma$  generated by the  $\lambda$  sets

$$\Gamma_{\geq t^*} = \{t \in \Gamma : t^* <_L t\}.$$

We can now see that the sets  $u_t^a$ , the numbers  $n_t$  and the filter  $D$  satisfy conditions (i)-(iv) of Lemma 4 with  $L$  instead of  $\lambda^+$ : Conditions (i) and (ii)



are trivial in this case. Condition (iii) is verified as follows: Suppose  $B$  is finite. Let  $a \in L$  be such that  $(\forall x \in B)(x <_L a)$ . Let  $\vec{a}$  enumerate  $B \cup \{a\}$  in increasing order and let  $t^* = tp(\vec{a})$ . Clearly

$$t \in \Gamma_{\geq t^*} \Rightarrow B \subseteq u_t^a.$$

Condition (iv) follows directly from Definition 9 and Proposition 8.

To get the Lemma on  $\lambda^+$  we observe that since  $L$  is  $\lambda^+$ -like, we can assume that  $\langle \lambda^+, < \rangle$  is a submodel of  $\langle L, <_L \rangle$ . Then we define  $v_t^\alpha = u_t^\alpha \cap \{\beta : \beta < \alpha\}$ . Conditions (i)-(iv) of Lemma 5 are still satisfied. Also having  $D$  a filter of  $\Gamma$  instead of  $\lambda$  is immaterial as  $|\Gamma| = \lambda$ .  $\square$

Now back to the proof of Lemma 4. Suppose  $u_i^\zeta, n_i$  and  $D$  are as in Lemma 5, and suppose  $D'$  is an arbitrary regular filter on  $\lambda$ . Let  $\{A_\alpha : \alpha < \lambda\}$  be a family of sets witnessing the regularity of  $D'$ , and let  $\{Z_\alpha : \alpha < \lambda\}$  be the family generating  $D$ . We define a function  $h : \lambda \rightarrow \lambda$  as follows. Suppose  $i < \lambda$ . Then let

$$h(i) \in \bigcap \{Z_\alpha \mid i \in A_\alpha\}.$$

Now define  $v_\alpha^\zeta = u_{h(\alpha)}^\zeta$ . Define also  $n_\alpha = n_{h(\alpha)}$ . Now the sets  $v_\alpha^\zeta$  and the numbers  $n_\alpha$  satisfy the conditions of Lemma 4.  $\square$

## 4 Is $\square_\lambda^{b^*}$ needed for Lemma 5?

In this section we show that the conclusion of Lemma 5 (and hence of Lemma 4) implies  $\square_\lambda^{b^*}$  for singular strong limit  $\lambda$ . By [11, Theorem 2.3 and Remark 2.5],  $\square_\lambda^{b^*}$  is equivalent, for singular strong limit  $\lambda$ , to the following principle:

$\mathcal{S}_\lambda$  : There are sets  $\langle C_a^i : a < \lambda^+, i < cf(\lambda) \rangle$  such that

- (i) If  $i < j$ , then  $C_a^i \subseteq C_a^j$ .
- (ii)  $\bigcup_i C_a^i = a$ .
- (iii) If  $b \in C_a^i$ , then  $C_b^i = C_a^i \cap b$ .
- (iv)  $\sup\{otp(C_a^i) : a < \lambda^+\} < \lambda$ .

Thus it suffices to prove:

**Proposition 10** *Suppose the sets  $u_i^\zeta$  and the filter  $D$  are as given by Lemma 5 and  $\lambda$  is a limit cardinal. Then  $\mathcal{S}_\lambda$  holds.*

**Proof.** Suppose  $\mathcal{A} = \{A_\alpha : \alpha < \lambda\}$  is a family of sets generating  $D$ . W.l.o.g.,  $\mathcal{A}$  is closed under finite intersections. Let  $\lambda$  be the union of the increasing sequence  $\langle \lambda_\alpha : \alpha < cf(\lambda) \rangle$ , where  $\lambda_0 \geq \omega$ . Let the sequence  $\langle \Gamma_\alpha : \alpha < cf(\lambda) \rangle$  satisfy:

- (a)  $|\Gamma_\alpha| \leq \lambda_\alpha$
- (b)  $\Gamma_\alpha$  is continuously increasing in  $\alpha$  with  $\lambda$  as union
- (c) If  $\beta_1, \dots, \beta_n \in \Gamma_\alpha$ , then there is  $\gamma \in \Gamma_\alpha$  such that

$$A_\gamma = A_{\beta_1} \cap \dots \cap A_{\beta_n}.$$

The sequence  $\langle \Gamma_\alpha : \alpha < cf(\lambda) \rangle$  enables us to define a sequence that will witness  $\mathcal{S}_\lambda$ . For  $\alpha < cf(\lambda)$  and  $\zeta < \lambda^+$ , let

$$V_\zeta^\alpha = \{\xi < \zeta : (\exists \gamma \in \Gamma_\alpha)(A_\gamma \subseteq \{i : \xi \in u_i^\zeta\})\}.$$

**Lemma 11** (1)  $\langle V_\zeta^\alpha : \alpha < \lambda \rangle$  is a continuously increasing sequence of subsets of  $\zeta$ ,  $|V_\zeta^\alpha| \leq \lambda_\alpha$ , and  $\bigcup \{V_\zeta^\alpha : \alpha < cf(\lambda)\} = \zeta$ .

(2) If  $\xi \in V_\zeta^\alpha$ , then  $V_\xi^\alpha = V_\zeta^\alpha \cap \xi$ .

**Proof.** (1) is a direct consequence of the definitions. (2) follows from the respective property of the sets  $u_i^\zeta$ .  $\square$

**Lemma 12**  $\sup\{otp(V_\zeta^\alpha) : \zeta < \lambda^+\} \leq \lambda_\alpha^+$ .

**Proof.** By the previous Lemma,  $|V_\zeta^\alpha| \leq \lambda_\alpha$ . Therefore  $otp(V_\zeta^\alpha) < \lambda_\alpha^+$  and the claim follows.  $\square$

The proof of the proposition is complete: (i)-(iii) follows from Lemma 11, (iv) follows from Lemma 12 and the assumption that  $\lambda$  is a limit cardinal.  $\square$

More equivalent conditions for the case  $\lambda$  singular strong limit,  $D$  a regular ultrafilter on  $\lambda$ , are under preparation.

## 5 Ehrenfeucht-Fraïssé-games

Let  $M$  and  $N$  be two first order structures of the same vocabulary  $L$ . All vocabularies are assumed to be relational. The *Ehrenfeucht-Fraïssé-game of length  $\gamma$  of  $M$  and  $N$*  denoted by  $\text{EFG}_\gamma$  is defined as follows: There are two players called I and II. First I plays  $x_0$  and then II plays  $y_0$ . After this I plays  $x_1$ , and II plays  $y_1$ , and so on. If  $\langle (x_\beta, y_\beta) : \beta < \alpha \rangle$  has been played and  $\alpha < \gamma$ , then I plays  $x_\alpha$  after which II plays  $y_\alpha$ . Eventually a sequence  $\langle (x_\beta, y_\beta) : \beta < \gamma \rangle$  has been played. The rules of the game say that both players have to play elements of  $M \cup N$ . Moreover, if I plays his  $x_\beta$  in  $M$  ( $N$ ), then II has to play his  $y_\beta$  in  $N$  ( $M$ ). Thus the sequence  $\langle (x_\beta, y_\beta) : \beta < \gamma \rangle$  determines a relation  $\pi \subseteq M \times N$ . Player II wins this round of the game if  $\pi$  is a partial isomorphism. Otherwise I wins. The notion of winning strategy is defined in the usual manner. We say that a player *wins*  $\text{EFG}_\gamma$  if he has a winning strategy in  $\text{EFG}_\gamma$ .

Note that if II has a winning strategy in  $\text{EFG}_\gamma$  on  $M$  and  $N$ , where  $M$  and  $N$  are of size  $\leq |\gamma|$ , then  $M \cong N$ .

Assume  $L$  is of cardinality  $\leq \lambda$  and for each  $i < \lambda$  let  $M_i$  and  $N_i$  are elementarily equivalent  $L$ -structures. Shelah proved in [12] that if  $D$  is a regular filter on  $\lambda$ , then Player II has a winning strategy in the game  $\text{EFG}_\gamma$  on  $\prod_i M_i/D$  and  $\prod_i N_i/D$  for each  $\gamma < \lambda^+$ . We show that under a stronger assumption, II has a winning strategy even in the game  $\text{EFG}_{\lambda^+}$ . This makes a big difference because, assuming the models  $M_i$  and  $N_i$  are of size  $\leq \lambda^+$ ,  $2^\lambda = \lambda^+$ , and the models  $\prod_i M_i/D$  and  $\prod_i N_i/D$  are of size  $\leq \lambda^+$ . Then by the remark above, if II has a winning strategy in  $\text{EFG}_{\lambda^+}$ , the reduced powers are actually isomorphic. Hyttinen [4] proved this under the assumption that the filter is, in his terminology, semigood.

**Theorem 13** *Assume  $\langle \aleph_0, \aleph_1 \rangle \rightarrow \langle \lambda, \lambda^+ \rangle$ . Let  $L$  be a vocabulary of cardinality  $\leq \lambda$  and for each  $i < \lambda$  let  $M_i$  and  $N_i$  be two elementarily equivalent  $L$ -structures. If  $D$  is a regular filter on  $\lambda$ , then Player II has a winning strategy in the game  $\text{EFG}_{\lambda^+}$  on  $\prod_i M_i/D$  and  $\prod_i N_i/D$ .*

**Proof.** We use Lemma 4. If  $i < \lambda$ , then, since  $M_i$  and  $N_i$  are elementarily equivalent, Player II has a winning strategy  $\sigma_i$  in the game  $\text{EFG}_{n_i}$  on  $M_i$  and  $N_i$ . We will use the set  $u_i^\zeta$  to put these short winning strategies together into one long winning strategy.

A “good” position is a sequence  $\langle (f_\zeta, g_\zeta) : \zeta < \xi \rangle$ , where  $\xi < \lambda^+$ , and for all  $\zeta < \xi$  we have  $f_\zeta \in \prod_i M_i$ ,  $g_\zeta \in \prod_i N_i$ , and if  $i < \lambda$ , then  $\langle (f_\epsilon(i), g_\epsilon(i)) : \epsilon \in u_i^\zeta \cup \{\zeta\} \rangle$  is a play according to  $\sigma_i$ .

Note that in a good position the equivalence classes of the functions  $f_\zeta$  and  $g_\zeta$  determine a partial isomorphism of the reduced products. The strategy of player II is to keep the position of the game “good”, and thereby win the game. Suppose  $\xi$  rounds have been played and II has been able to keep the position “good”. Then player I plays  $f_\xi$ . We show that player II can play  $g_\xi$  so that  $\langle (f_\zeta, g_\zeta) : \zeta \leq \xi \rangle$  remains “good”. Let  $i < \lambda$ . Let us look at  $\langle (f_\epsilon(i), g_\epsilon(i)) : \epsilon \in u_i^\xi \rangle$ . We know that this is a play according to the strategy  $\sigma_i$  and  $|u_i^\xi| < n_i$ . Thus we can play one more move in  $EF_{n_i}$  on  $M_i$  and  $N_i$  with player I playing  $f_\xi(i)$ . Let  $g_\xi(i)$  be the answer of II in this game according to  $\sigma_i$ . The values  $g_\xi(i)$ ,  $i < \lambda$ , constitute the function  $g_\xi$ . We have showed that II can maintain a “good” position.  $\square$

**Corollary 14** *Assume GCH and  $\lambda$  regular (or just  $\langle \aleph_0, \aleph_1 \rangle \rightarrow \langle \lambda, \lambda^+ \rangle$  and  $2^\lambda = \lambda^+$ ). Let  $L$  be a vocabulary of cardinality  $\leq \lambda$  and for each  $i < \lambda$  let  $M_i$  and  $N_i$  be two elementarily equivalent  $L$ -structures. If  $D$  is a regular filter on  $\lambda$ , then  $\prod_i M_i/D \cong \prod_i N_i/D$ .*

## References

- [1] M. Benda, On reduced products and filters. Ann.Math.Logic 4 (1972), 1-29.
- [2] C. C. Chang, , A note on the two cardinal problem, Proc. Amer. Math. Soc., 16, 1965, 1148–1155,
- [3] C.C. Chang and J.Keisler, Model Theory, North-Holland.
- [4] T. Hyttinen, On  $\kappa$ -complete reduced products, Arch. Math. Logic, Archive for Mathematical Logic, 31, 1992, 3, 193–199
- [5] R. Jensen, The fine structure of the constructible hierarchy, With a section by Jack Silver, Ann. Math. Logic, 4, 1972, 229–308
- [6] B. Jónsson and P. Olin, Almost direct products and saturation, Compositio Math., 20, 1968, 125–132

- [7] J. Keisler, Ultraproducts and saturated models. Nederl.Akad.Wetensch. Proc. Ser. A 67 (=Indag. Math. 26) (1964), 178-186.
- [8] J. Kennedy and S. Shelah, On embedding models of arithmetic of cardinality  $\aleph_1$  into reduced powers, to appear.
- [9] W. Mitchell, Aronszajn trees and the independence of the transfer property, Ann. Math. Logic, 5, 1972/73, 21–46
- [10] S. Shelah, For what filters is every reduced product saturated?, Israel J. Math., 12, 1972, 23–31
- [11] S. Shelah, “Gap 1” two-cardinal principles and the omitting types theorem for  $L(Q)$ . Israel Journal of Mathematics vol 65 no. 2, 1989, 133–152.
- [12] S. Shelah, Classification theory and the number of non-isomorphic models, Second, North-Holland Publishing Co., Amsterdam, 1990, xxxiv+705